

## Zero-size objects in Riemann-Cartan spacetime

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**ABSTRACT:** We use the conservation law of the stress-energy and spin tensors to study the motion of massive zero-size objects in Riemann-Cartan geometry. The resultant world line equations turn out to exhibit a novel spin-curvature coupling. In particular, the spin of the Dirac particle does not couple to the background curvature. This is a consequence of its truly zero size which consistently rules out the orbital degrees of freedom. As a test of consistency, the wave packet solution of the free Dirac equation is considered. It is shown that the wave packet spin and orbital angular momentum disappear simultaneously in the zero-size limit.

**KEYWORDS:** Classical Theories of Gravity, Solitons Monopoles and Instantons.

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## 1. Introduction

The problem of particle motion in backgrounds of nontrivial geometry is usually addressed by using some form of the Mathisson-Papapetrou method [1, 2]. One starts with the covariant conservation law of the stress-energy and spin tensors of matter fields, and analyzes it under the assumption that matter is highly localized. In the lowest, single-pole approximation, the moving matter is viewed as a point particle. In the pole-dipole approximation, its non-zero size is taken into account.

The results found in literature can be summarized as follows. Spinless particles in the single-pole approximation obey the geodesic equation. In the pole-dipole approximation, the rotational angular momentum of the localized matter couples to spacetime curvature, and produces geodesic deviations [1–5]. If the particles have spin, the curvature couples to the total angular momentum, and the torsion to the spin alone [6–10].

What we are interested in is a consistent single-pole analysis of spinning particles in spacetimes with curvature and torsion. This is motivated by the observation that single-pole approximation eliminates the influence of particle thickness, and allows the derivation of the pure spin-curvature coupling. In fact, this is the only way to see the influence of curvature on the spin part of the total angular momentum. The ambiguous algebraic decomposition of the total angular momentum into spin and orbital contributions are of no help. What we need is a truly zero-size object. As it turns out, the existing literature on the subject does not have this sort of prediction.

The results that we have obtained are summarized as follows. Trajectories of spinning zero-size massive particles generally deviate from the geodesic lines. The deviation is due to the spin-curvature and spin-torsion couplings. These turn out to be different from what has been believed so far. In particular, the spin of the Dirac point particle does not couple to the curvature. If it is viewed as a wave packet solution of the Dirac equation, it does not couple to the torsion either. In fact, the wave packet spin and orbital angular momentum

disappear simultaneously in the zero-size limit. We can say that Dirac point particles behave as spinless objects.

The layout of the paper is as follows. In section 2, we define the conservation law of the stress-energy and spin tensors, and introduce the necessary geometric notions. The algebraic part of the conservation equations is solved in terms of the independent variables — the spin tensor and the symmetric part of the stress-energy tensor. After the brief recapitulation of the covariant multipole formalism, we define the single-pole approximation for the independent variables, only. Section 3 is devoted to the derivation of the particle world line equations. The actual derivation is only sketched, as the method has already been analyzed in detail in [12]. The resulting equations of motion are compared to the pole-dipole equations found in literature [8, 9]. As it turns out, they coincide up to a constraint that fixes the form of the spin tensor. This constraint is a consequence of our single-pole approximation, and has striking consequences on the dynamics of the Dirac particle. In section 4, we discuss the important case of totally antisymmetric spin tensor, and obtain a surprising result that spin of the Dirac particle does not couple to the background curvature. To check the consistency of our single-pole approximation, the wave packet solution of the free Dirac equation is analyzed. It is demonstrated that the wave packet spin and orbital angular momentum disappear simultaneously in the zero-size limit. In section 5, we give our final remarks.

Conventions in this paper are the following. Greek indices from the middle of the alphabet,  $\mu, \nu, \dots$ , are the spacetime indices, and run over  $0, 1, 2, 3$ . The indices from the beginning of the Greek alphabet,  $\alpha, \beta, \dots$ , take values  $1, 2, 3$ . The spacetime coordinates are denoted by  $x^\mu$ , the generic metric is denoted by  $g_{\mu\nu}(x)$ , and  $\eta_{\mu\nu}$  stands for the Minkowski metric. The signature convention is  $(-+++)$ .

## 2. The single-pole approximation

We begin with the covariant conservation of the fundamental matter currents — stress-energy tensor  $\tau^\mu{}_\nu$ , and spin tensor  $\sigma^\lambda{}_{\mu\nu}$ :

$$\left(D_\nu + \mathcal{T}^\lambda{}_{\nu\lambda}\right) \tau^\nu{}_\mu = \tau^\nu{}_\rho \mathcal{T}^\rho{}_{\mu\nu} + \frac{1}{2} \sigma^{\nu\rho\sigma} \mathcal{R}_{\rho\sigma\mu\nu}, \quad (2.1a)$$

$$\left(D_\nu + \mathcal{T}^\lambda{}_{\nu\lambda}\right) \sigma^\nu{}_{\rho\sigma} = \tau_{\rho\sigma} - \tau_{\sigma\rho}. \quad (2.1b)$$

Here,  $D_\nu$  is the covariant derivative with the nonsymmetric connection  $\Gamma^\lambda{}_{\mu\nu}$ , which acts on a vector  $v^\mu$  according to the rule  $D_\nu v^\mu \equiv \partial_\nu v^\mu + \Gamma^\mu{}_{\lambda\nu} v^\lambda$ . The torsion  $\mathcal{T}^\lambda{}_{\mu\nu}$ , and curvature  $\mathcal{R}^\mu{}_{\nu\rho\sigma}$  are defined in the standard way:

$$\mathcal{T}^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}, \quad \mathcal{R}^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\lambda\rho} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\mu{}_{\lambda\sigma} \Gamma^\lambda{}_{\nu\rho}.$$

The derivative  $D_\lambda$  is assumed to satisfy the metricity condition,  $D_\lambda g_{\mu\nu} = 0$ . As a consequence, the connection  $\Gamma^\lambda{}_{\mu\nu}$  is split into the Levi-Civita connection  $\{\overset{\lambda}{\mu\nu}\}$ , and the contorsion  $K^\lambda{}_{\mu\nu}$ :

$$\Gamma^\lambda{}_{\mu\nu} = \{\overset{\lambda}{\mu\nu}\} + K^\lambda{}_{\mu\nu}, \quad K^\lambda{}_{\mu\nu} \equiv -\frac{1}{2} \left( \mathcal{T}^\lambda{}_{\mu\nu} - \mathcal{T}^\lambda{}_{\nu\mu} + \mathcal{T}^\lambda{}_{\mu\nu} \right).$$

We shall also introduce the Riemannian covariant derivative  $\nabla_\mu \equiv D_\mu(\Gamma \rightarrow \{\})$ , and the Riemannian curvature tensor  $R^\mu{}_{\nu\rho\sigma} \equiv \mathcal{R}^\mu{}_{\nu\rho\sigma}(\Gamma \rightarrow \{\})$ . The relation connecting the two curvature tensors reads:

$$\mathcal{R}^\mu{}_{\nu\lambda\rho} = R^\mu{}_{\nu\lambda\rho} + 2\nabla_{[\lambda}K^\mu{}_{\nu\rho]} + 2K^\mu{}_{\sigma[\lambda}K^\sigma{}_{\nu\rho]},$$

where the indices in square brackets are antisymmetrized.

Given the system of conservation equations (2.1), one finds that the second one has no dynamical content. Indeed, the antisymmetric part of stress-energy tensor is completely determined by the spin tensor. One can use (2.1b) to eliminate  $\tau^{[\mu\nu]}$  from the equation (2.1a), and thus obtain the conservation equation, in which only  $\tau^{(\mu\nu)}$  and  $\sigma^{\lambda\mu\nu}$  components appear. The resulting equation reads:

$$\nabla_\nu \left( \tau^{(\mu\nu)} + \frac{1}{2}K_{\lambda\rho}{}^\mu \sigma^{\nu\lambda\rho} - K^{[\mu}{}_{\lambda\rho} \sigma^{\rho\lambda\nu]} - \nabla_\rho \sigma^{(\mu\nu)\rho} \right) = \frac{1}{2}\sigma_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu}. \quad (2.2)$$

This will be the starting point of the derivation of the particle world line equations.

Let us now introduce the multipole formalism, which is necessary for the derivation. It has been shown in refs. [11, 12] that an exponentially decreasing function can be expanded into a series of  $\delta$ -function derivatives. For example, a scalar  $V(x)$ , well localized around the line  $\mathcal{M}$ , can be written in a manifestly covariant way as

$$V(x) = \int_{\mathcal{M}} d\tau \left[ M(\tau) \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} - \nabla_\rho \left( M^\rho(\tau) \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} \right) + \dots \right]. \quad (2.3)$$

Here,  $\mathcal{M}$  is a timelike line  $x^\mu = z^\mu(\tau)$  parametrized by the proper distance,  $d\tau^2 = g_{\mu\nu}dz^\mu dz^\nu$ , and the coefficients  $M(\tau)$ ,  $M^\rho(\tau)$ , ... are spacetime tensors called multipole coefficients. It has been shown in ref. [12] that one may truncate the series in a covariant way in order to approximate the description of matter. Truncation after the leading term is called *single-pole* approximation, truncation after the second term is called *pole-dipole* approximation. The physical interpretation of these approximations is the following. In the single-pole approximation, one assumes that the particle has no thickness, which means that matter is localized in a point. All higher approximations, including pole-dipole, allow for the nonzero thickness, and thus, for the nontrivial internal motion.

Apart from being covariant with respect to diffeomorphisms, the series (2.3) possesses two extra gauge symmetries. The first is a consequence of the fact that there are redundant coefficients in this decomposition. Indeed, only three out of four  $\delta$ -functions in each term of the multipole expansion (2.3) are effective in modeling particle trajectory in 4-dimensional spacetime. The extra  $\delta$ -function and the extra integration are introduced only to covariantize the expressions. The derivatives parallel to the world line are integrated out, as they should, considering the fact that matter is not localized in time. As a consequence, the parallel components of the multipole coefficients  $M^\rho$ ,  $M^{\rho\lambda}$ , ... effectively disappear. It has been shown in ref. [12] that the corresponding gauge symmetry, named *extra symmetry 1*, in the pole-dipole approximation reads:

$$\delta_1 M = \nabla \epsilon, \quad \delta_1 M^\rho = u^\rho \epsilon.$$

Here,  $u^\mu \equiv dz^\mu/d\tau$  is the particle 4-velocity,  $\epsilon(\tau)$  is a gauge parameter, and  $\nabla$  stands for the Riemannian covariant derivative along the particle trajectory ( $\nabla v^\mu = dv^\mu/d\tau + \{\lambda^\mu_{\lambda\rho}\}v^\lambda u^\rho$ ). We see that the parallel component of  $M^\rho$  transforms as  $\delta_1(M^\rho u_\rho) = -\epsilon$ , and can be gauged away. In fact, one can show that the parallel components of the higher multipoles are also pure gauge. In the gauge fixed multipole expansion, the only derivatives that appear are those orthogonal to the world line.

The second extra symmetry stems from the fact that the choice of the line  $x^\mu = z^\mu(\tau)$  in the expansion (2.3) is arbitrary. If we use another line, let us say  $x^\mu = z'^\mu(\tau)$ , the coefficients  $M, M^\rho, \dots$  will change to  $M', M'^\rho, \dots$  while leaving the scalar function  $V(x)$  invariant. The transformation law of the  $M$ -coefficients, generated by the replacement  $z^\mu \rightarrow z'^\mu$ , defines the gauge symmetry that we call *extra symmetry 2*.

The extra symmetry 2 is an exact symmetry of the full expansion (2.3), but only approximate symmetry of the truncated series. In the pole-dipole approximation, it has the form

$$\delta_2 z^\mu = \epsilon^\mu, \quad \delta_2 M = M u_\rho \nabla \epsilon^\rho, \quad \delta_2 M^\rho = -M \epsilon^\rho,$$

provided the  $M$ -coefficients are subject to the hierarchy

$$M = \mathcal{O}_0, \quad M^\rho = \mathcal{O}_1, \quad M^{\rho\lambda} = \mathcal{O}_2, \dots,$$

and the free parameters  $\epsilon^\mu(\tau)$  satisfy  $\epsilon^\mu = \mathcal{O}_1$ . Here,  $\mathcal{O}_n$  stands for the order of smallness, and the condition  $\epsilon^\mu = \mathcal{O}_1$  ensures that the order of truncation is not violated by the action of the symmetry transformations [12]. In the pole-dipole and higher approximations, fixing the gauge of extra symmetry 2 defines the particle centre of mass. In the single-pole approximation, the extra symmetry 2 is trivial.

Now, we shall replace the general function  $V(x)$  with the stress-energy and spin tensors of the localized matter. In order to describe a strict point particle, we choose  $\tau^{(\mu\nu)}$  and  $\sigma^{\lambda\mu\nu}$  in the form

$$\tau^{(\mu\nu)} = \int_{\mathcal{M}} d\tau b^{\mu\nu}(\tau) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}}, \tag{2.4a}$$

$$\sigma^{\lambda\mu\nu} = \int_{\mathcal{M}} d\tau c^{\lambda\mu\nu}(\tau) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}}, \tag{2.4b}$$

where  $b^{\mu\nu}(\tau)$  and  $c^{\lambda\mu\nu}(\tau)$  are the corresponding multipole coefficients. We emphasize here that this is not how single-pole approximation is defined in the existing literature [8, 9]. There, the antisymmetric part of stress-energy tensor  $\tau^{\mu\nu}$  has also been treated in the single-pole manner. As  $\tau^{[\mu\nu]}$  is not an independent variable, this imposed unnecessary constraints on  $\sigma^{\lambda\mu\nu}$ . In particular, the spin of the Dirac particle was ruled out. To overcome this problem, the authors of ref. [9] abandoned single-pole in favour of pole-dipole approximation. Their subsequent limit of vanishing orbital angular momentum should have brought them back to the single-pole regime. In what follows, however, we shall demonstrate that it is not quite so, and that such a limit is not equivalent to the single-pole approximation as defined in (2.4).

### 3. Equations of motion

The particle equations of motion are derived in the following way. We insert (2.4) into (2.2), and solve for the unknown variables  $z(\tau)$ ,  $b^{\mu\nu}(\tau)$  and  $c^{\lambda\mu\nu}(\tau)$ . The algorithm for solving this type of equation is discussed in detail in [11, 12], and here we only sketch it. The first step is to multiply the equation (2.2) with an arbitrary spacetime function  $f_\mu(x)$ , and integrate over the spacetime. The resulting equation depends on the function  $f_\mu$  and its first and second covariant derivatives, evaluated on the line  $x^\mu = z^\mu(\tau)$ :

$$\int d\tau \left[ c^{(\mu\nu)\rho} f_{\mu;\nu\rho} + \left( b^{\mu\nu} - K^{[\mu}{}_{\lambda\rho} c^{\rho\lambda\nu]} + \frac{1}{2} K_{\lambda\rho}{}^\mu c^{\nu\lambda\rho} \right) f_{\mu;\nu} + \frac{1}{2} c_{\nu\rho\lambda} \left( \nabla^\mu K^{\rho\lambda\nu} \right) f_\mu \right] = 0,$$

where  $f_{\mu;\nu} \equiv (\nabla_\nu f_\mu)_{x=z}$ ,  $f_{\mu;\nu\rho} \equiv (\nabla_\rho \nabla_\nu f_\mu)_{x=z}$ . Owing to the arbitrariness of the function  $f_\mu(x)$ , the terms proportional to its independent derivatives separately vanish. To find the independent derivatives of the test function  $f_\mu$ , we make use of the particle 4-velocity  $u^\mu \equiv dz^\mu/d\tau$ , and the Riemannian covariant derivative along the particle trajectory  $\nabla$ . The 4-velocity  $u^\mu$  is normalized as  $u^\mu u_\mu = -1$ , and the action of  $\nabla$  on a vector field  $v^\mu(\tau)$  is defined by  $\nabla v^\mu \equiv dv^\mu/d\tau + \left\{ \begin{smallmatrix} \mu \\ \lambda\rho \end{smallmatrix} \right\} v^\lambda u^\rho$ . Next, we decompose the derivatives of the vector field  $f_\mu(x)$  into components orthogonal and parallel to the world line  $x^\mu = z^\mu(\tau)$ :

$$f_{\mu;\lambda} = f_{\mu\lambda}^\perp - u_\lambda \nabla f_\mu, \quad (3.1a)$$

$$f_{\mu;(\lambda\rho)} = f_{\mu\lambda\rho}^\perp - 2h_{\mu(\lambda}^\perp u_{\rho)} + h_\mu u_\lambda u_\rho, \quad (3.1b)$$

$$f_{\mu;[\lambda\rho]} = \frac{1}{2} R^\sigma{}_{\mu\lambda\rho} f_{\sigma}. \quad (3.1c)$$

Here, the orthogonal and parallel components are obtained by using the projectors

$$P_{\perp\nu}^\mu = \delta_\nu^\mu + u^\mu u_\nu, \quad P_{\parallel\nu}^\mu = -u^\mu u_\nu. \quad (3.2)$$

More precisely,  $f_{\mu\lambda}^\perp = P_{\perp\lambda}^\sigma f_{\mu;\sigma}$ ,  $f_{\mu\lambda\rho}^\perp = P_{\perp\lambda}^\sigma P_{\perp\rho}^\nu f_{\mu;(\sigma\nu)}$ ,  $h_{\mu\lambda}^\perp = P_{\perp\lambda}^\sigma u^\nu f_{\mu;(\sigma\nu)}$  and  $h_\mu = u^\sigma u^\nu f_{\mu;(\sigma\nu)}$ . Direct calculation yields

$$\begin{aligned} h_\mu &= \nabla \nabla f_\mu - (\nabla u^\nu) f_{\mu\nu}^\perp, \\ h_{\mu\rho}^\perp &= P_{\perp\rho}^\nu \nabla f_{\mu\nu}^\perp - (\nabla u_\rho) \nabla f_\mu + \frac{1}{2} P_{\perp\rho}^\lambda u^\nu R^\sigma{}_{\mu\nu\lambda} f_\sigma, \end{aligned} \quad (3.3)$$

which tells us that the only independent derivatives on the line  $x^\mu = z^\mu(\tau)$  are  $f_\mu$ ,  $f_{\mu\nu}^\perp$  and  $f_{\mu\nu\rho}^\perp$ . We can now use (3.1) and (3.3) to group the coefficients in terms proportional to the independent derivatives of  $f_\mu$ . The obtained equation has the following general structure:

$$\int_{\mathcal{M}} d\tau \left[ X^{\mu\nu\rho} f_{\mu\nu\rho}^\perp + X^{\mu\nu} f_{\mu\nu}^\perp + X^\mu f_\mu \right] = 0,$$

where  $X^{\mu\nu\rho}$ ,  $X^{\mu\nu}$  and  $X^\mu$  are composed of various combinations of multipole coefficients  $b^{\mu\nu}$  and  $c^{\lambda\mu\nu}$ , external fields  $K^\lambda{}_{\mu\nu}$  and  $R^\mu{}_{\nu\rho\sigma}$ , and their derivatives. In all the expressions, the external fields are evaluated on the world line  $x^\mu = z^\mu(\tau)$ . Owing to the fact that  $f_\mu$ ,  $f_{\mu\nu}^\perp$  and  $f_{\mu\nu\rho}^\perp$  are independent functions on the world line, we deduce that the  $X$ -terms must separately vanish. The equation  $X^{\mu\nu\rho} = 0$  has a simple algebraic form

$$P_{\perp\lambda}^\mu P_{\perp\sigma}^\nu c^{(\lambda\sigma)\rho} = 0,$$

which is easily solved for  $c^{\lambda\mu\nu}$ . This yields

$$c^{\lambda\mu\nu} = 2u^\lambda s^{\mu\nu} + s^{\lambda\mu\nu},$$

where  $s^{\mu\nu} \equiv -s^{\nu\mu}$  and  $s^{\lambda\mu\nu} \equiv -s^{\lambda\nu\mu} \equiv s^{\nu\lambda\mu}$  are totally antisymmetric, but otherwise free parameters. The equations  $X^{\mu\nu} = 0$  and  $X^\mu = 0$  are much more complicated. The procedure goes as follows. First, we use the above decomposition of  $c^{\lambda\mu\nu}$  to perform a similar split of the  $b^{\mu\nu}$  coefficients. A new free parameter  $m(\tau)$  appears to characterize the leading term of  $b^{\mu\nu}$ . Then, the equations  $X^{\mu\nu} = 0$  and  $X^\mu = 0$  are rewritten in terms of the undetermined parameters  $m$ ,  $s^{\mu\nu}$  and  $s^{\lambda\mu\nu}$ , and properly rearranged. Skipping the details of the diagonalization procedure, which has thoroughly been demonstrated in ref. [12], we display the final result:

- world line equation

$$\nabla \left[ m u^\mu + 2u_\rho (\nabla s^{\mu\rho} + D^{\mu\rho}) \right] - u^\nu s^{\rho\sigma} R^\mu{}_{\nu\rho\sigma} = \frac{1}{2} c_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu}, \quad (3.4a)$$

- spin precession equation

$$P_{\perp\rho}{}^\mu P_{\perp\sigma}{}^\nu (\nabla s^{\rho\sigma} + D^{\rho\sigma}) = 0, \quad (3.4b)$$

- stress-energy coefficients

$$b^{\mu\nu} = m u^\mu u^\nu + 2u_\lambda u^{(\mu} \nabla s^{\nu)\lambda} - \frac{1}{2} K_{\lambda\rho}{}^{(\mu} c^{\nu)\lambda\rho}, \quad (3.4c)$$

- spin tensor coefficients

$$c^{\lambda\mu\nu} = 2u^\lambda s^{\mu\nu} + s^{\lambda\mu\nu}. \quad (3.4d)$$

In these equations, the scalar  $m(\tau)$ , and the totally antisymmetric tensors  $s^{\mu\nu}(\tau)$  and  $s^{\lambda\mu\nu}(\tau)$  are free parameters. They determine the stress-energy and spin tensors via (3.4c) and (3.4d). The shorthand notation

$$D^{\mu\nu} \equiv K^{[\mu}{}_{\lambda\rho} c^{\rho\lambda\nu]} + \frac{1}{2} K_{\lambda\rho}{}^{[\mu} c^{\nu]\rho\lambda}$$

is introduced to simplify the cumbersome expressions.

The obtained single-pole equations differ from the known pole-dipole result [6–10] by the presence of the constraint (3.4d). It is a consequence of our assumption that the particle has no thickness, and therefore no orbital degrees of freedom. In the existing literature, an analogous but more restrictive constraint appears in this regime [7, 9]. This is because the antisymmetric part of the stress-energy tensor  $\tau^{[\mu\nu]}$  has been treated as an independent variable, in spite of the restriction (2.1b). In our approach, the only independent variables are  $\sigma^{\lambda\mu\nu}$  and  $\tau^{(\mu\nu)}$ , and the resulting constraint (3.4d) is not so strong. In particular, it does not rule out the free Dirac field, or any other massive elementary field. Indeed, the formula  $c^{\lambda\mu\nu} = u^\lambda s^{\mu\nu} + \frac{1}{s} u^{[\mu} s^{\nu]\lambda}$  for the spin tensor of the elementary particle of spin  $s$  (see ref. [10]) is a special case of (3.4d).

In what follows, we shall examine the special case of spin 1/2 pointlike matter. Surprisingly, we shall discover that spin 1/2 does not couple to the curvature, leading to geodesic trajectories in torsionless spacetimes.

#### 4. The Dirac particle

The basic property of Dirac matter is the total antisymmetry of its spin tensor  $\sigma^{\lambda\mu\nu}$ . As a consequence, the coefficients  $c^{\lambda\mu\nu}$  are also totally antisymmetric, and the constraint (3.4d) implies

$$s^{\mu\nu} = 0. \tag{4.1}$$

The vanishing of the  $s^{\mu\nu}$  component of the spin tensor has far reaching consequences. First, we see that the spin-curvature and spin-orbit couplings disappear from the world line equation (3.4a). Second, the spin precession equation (3.4b) becomes a constraint equation. If we define the spin vector  $s^\mu$  by  $s^{\mu\nu\rho} \equiv e^{\mu\nu\rho\lambda} s_\lambda$ , and the axial component of the contorsion  $K^\mu$  as  $K^\mu \equiv e^{\mu\nu\rho\lambda} K_{\nu\rho\lambda}$ , where  $e^{\mu\nu\rho\sigma}$  is the covariant totally antisymmetric Levi-Civita tensor, the equations (3.4) become

$$\nabla \left( m u^\mu + K^{[\mu} s^{\nu]} u_\nu \right) + \frac{1}{2} s^\nu \nabla^\mu K_\nu = 0, \tag{4.2a}$$

$$K_{\perp}^{[\mu} s_{\perp}^{\nu]} = 0. \tag{4.2b}$$

As we can see, the spin couples only to the axial component of the contorsion, which means that *Dirac point particles follow geodesic trajectories in torsionless spacetimes*. At the same time, the absence of torsion trivializes the equation (4.2b), and no information on the behavior of the spin vector is available. If the background torsion has nontrivial axial component, a geodesic deviation appears, but also a very strong constraint on the spin vector. Indeed, the equation (4.2b) implies that the orthogonal component of  $s^\mu$  always orients itself along the background direction  $K_{\perp}^\mu$ . This unusual behavior suggests that the spin vector of the Dirac point particle might be zero, after all. In fact, the world line equations (3.4) are derived under very general assumptions of the existence of pointlike solutions in an arbitrary field theory. They do not care about peculiarities of specific theories or specific types of localized solutions. In what follows, we shall analyze the wave packet solutions of the flat space Dirac equation, with the idea to check if they can be viewed as point particles.

Let us construct an example. We start with the free Dirac Lagrangian

$$\mathcal{L} = \frac{i}{2} [\bar{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi} \gamma^\mu \psi)] - m \bar{\psi} \psi,$$

where Dirac  $\gamma$ -matrices satisfy the usual anticommutation relations  $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ , and are used in their conventional representation ( $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ ). Then, we construct a wave packet. The wave packet is a solution which is well localized in space, but resembles a plane wave inside. To be viewed as a particle, its size  $\ell$  is considered in the limit  $\ell \rightarrow 0$ . At the same time, the particle stability is achieved in the limit  $\lambda/\ell \rightarrow 0$ , where  $\lambda$  is its wavelength. We construct it as follows. At the initial moment  $x^0 = 0$ , we choose the configuration

$$\psi(\vec{r}, 0) \equiv A e^{-\frac{r^2}{\ell^2}} \psi_p(\vec{r}, 0),$$



where

$$\psi_p(x) \equiv \sqrt{\frac{k^0 + m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{k^3}{k^0 + m} \\ 0 \end{bmatrix} e^{ik_\mu x^\mu}$$

is the plane-wave solution of the Dirac equation  $(i\gamma^\mu \partial_\mu - m)\psi = 0$ . It propagates along the  $x^3$ -axis ( $k^1 = k^2 = 0$ ,  $k^0 \equiv \sqrt{m^2 + (k^3)^2}$ ), and is polarized upwards, for convenience. The exponential function  $\exp(-r^2/\ell^2)$  cuts out a small piece of the plane wave, and defines its size  $\ell$ , while  $A$  is the overall amplitude of the packet. The wavelength  $\lambda$  is proportional to  $1/|\vec{k}|$ . Using the Dirac equation, we can calculate time derivatives and thereby determine time evolution of this packet. In fact, we only need first time derivatives, as neither  $\tau^{\mu\nu}$  nor  $\sigma^{\lambda\mu\nu}$  depend on higher derivatives:

$$\tau_{\mu\nu} = i [\bar{\psi} \gamma_\mu \partial_\nu \psi - (\partial_\nu \bar{\psi}) \gamma_\mu \psi] - 2\eta_{\mu\nu} \mathcal{L}, \quad \sigma^{\lambda\mu\nu} = \varepsilon^{\lambda\mu\nu\rho} \bar{\psi} \gamma_5 \gamma_\rho \psi. \quad (4.3)$$

The wave packet expressions of these currents at  $x^0 = 0$  are obtained straightforwardly:

$$\tau^{(00)} = -2|A|^2 e^{-\frac{2r^2}{\ell^2}} \frac{(k^0)^2}{m}, \quad \tau^{(33)} = -2|A|^2 e^{-\frac{2r^2}{\ell^2}} \frac{(k^3)^2}{m}, \quad (4.4a)$$

$$\tau^{(0\alpha)} = -2|A|^2 e^{-\frac{2r^2}{\ell^2}} \frac{k^0}{m} \left( k^3 \eta^{3\alpha} - \frac{x_\beta}{\ell^2} \varepsilon^{\alpha\beta 3} \right), \quad (4.4b)$$

$$\sigma^{123} = -|A|^2 e^{-\frac{2r^2}{\ell^2}} \frac{k^3}{m}, \quad \sigma^{012} = -|A|^2 e^{-\frac{2r^2}{\ell^2}} \frac{k^0}{m}, \quad (4.4c)$$

where only non-vanishing components are displayed. Now, we want to rewrite the currents (4.4) as a series of  $\delta$ -function derivatives. We first fix diffeomorphisms by imposing the condition  $g_{\mu\nu} = \eta_{\mu\nu}$ , and extra symmetry 1 by keeping only spatial components of the M coefficients ( $M^0 = M^{0\rho} = \dots = 0$ ). The decomposition formula (2.3) is thereby reduced to

$$V(x) = \int_{\mathcal{M}} d\tau \left[ M \delta^{(4)}(x - z) - \partial_\alpha \left( M^\alpha \delta^{(4)}(x - z) \right) + \dots \right].$$

In general, the line  $x^\mu = z^\mu(\tau)$  is arbitrary, but the simplest expressions are obtained if it coincides with the wave packet trajectory. Thus, we choose  $z^1 = z^2 = 0$  in accordance with the fact that the packet propagates along the  $x^3$  axis. As for the  $z^3$  component, we do not need its full  $\tau$  dependence because we are only interested in the packet behaviour at  $x^0 = 0$ . There, the proper length  $\tau$  is chosen in accordance with  $z^0(0) = z^3(0) = 0$ , which is sufficient for the proper definition of the  $\delta$ -expansion at  $x^0 = 0$ .

The multipole coefficients are obtained by multiplying  $V(x)$  by a number of  $(x^\alpha - z^\alpha)$  factors, and integrating over the 3-space. In our example, the simple integration of (4.4) yields the monopole coefficients  $b^{\mu\nu}$  and  $c^{\lambda\mu\nu}$ , while multiplication with  $(x^\alpha - z^\alpha)$ , and subsequent integration gives the dipole coefficients  $b^{\mu\nu\alpha}$ . The resultant non-zero monopoles are

$$\begin{aligned} b^{00} &= a \ell^3 (k^0)^2, & b^{33} &= a \ell^3 (k^3)^2, & b^{03} &= a \ell^3 k^0 k^3, \\ c^{123} &= \frac{a}{2} \ell^3 k^3, & c^{012} &= \frac{a}{2} \ell^3 k^0, \end{aligned}$$

while there are only two non-zero dipoles,

$$b^{012} = -b^{021} = \frac{a}{4} \ell^3 k^0 .$$

The higher multipoles are of the order  $\ell^5(k^3)^2$  or higher. Here,  $a(k)$  is the overall factor whose explicit form is not needed in the subsequent discussion.

Now, we shall consider the single-pole limit  $\ell \rightarrow 0$ , while respecting the wave packet stability condition  $\lambda \ll \ell$ . First, we choose the overall amplitude  $a(k)$  in the form

$$a(k) \sim \frac{\lambda^2}{\ell^3} ,$$

thereby normalizing the monopole coefficients to be of the order of unity. Then, using the single-pole behavior  $k^3 \sim k^0 \sim \lambda^{-1}$ , we find

$$b^{00} \sim b^{33} \sim b^{03} \sim 1, \quad c^{123} \sim c^{012} \sim \lambda, \quad b^{012} \sim \lambda .$$

The higher multipoles are of the order  $\ell^2$ , and thus, neglected. This is a realization of the pole-dipole approximation. In the single-pole regime, however, only the lowest terms are retained in the limit  $\ell \rightarrow 0$ . This means that, respecting  $\lambda \ll \ell$ , the terms proportional to  $\lambda$  must also be dropped. As a result, the *spin monopole coefficients*  $c^{\lambda\mu\nu}$  *vanish simultaneously with the orbital dipole coefficients*  $b^{\mu\nu\lambda}$ .

The reason for this unusual behaviour is found in the constraint  $c^{012} = 2b^{012}$ . It is obtained by the integration of the more general relation

$$x^\alpha \tau^{(0\beta)} - x^\beta \tau^{(0\alpha)} = \sigma^{0\alpha\beta} + div \tag{4.5}$$

that is found to constrain the wave packet currents (4.4). It relates the wave packet spin to its orbital angular momentum, so that the expected disappearance of orbital degrees of freedom in the limit  $\ell \rightarrow 0$  is followed by the unexpected disappearance of the spin itself.

To summarize, we see that the spin vector  $s^\mu$  vanishes in the single-pole approximation, and the particle trajectory becomes a geodesic line even in the presence of torsion. The validity of this conclusion, however, demands some sort of equivalence principle to hold. This is because the considered wave packet is a solution of the free Dirac equation, and the inclusion of curvature or torsion may destroy it. What we can do is to consider weak gravity, so that terms quadratic in curvature and torsion are neglected. In that case, the free wave packets are a good approximation to the exact solution, which implies that *Dirac point particles behave as spinless objects in an external gravitational field*. They can still probe the spacetime curvature, but for the probe of the background torsion, one needs a thick particle.

## 5. Concluding remarks

In this paper, we have considered the motion of point particles with nonzero spin in spacetimes with curvature and torsion. Using the covariant multipole formalism developed in ref. [12], the world line equations are derived in the lowest, single-pole approximation. This

way, the particle thickness, and the corresponding internal motion, have been eliminated. Only mass and spin remained to characterize the particle internal structure.

In our approach, the single-pole behaviour has been adopted for truly independent variables only. In particular, the antisymmetric part of the stress-energy tensor has been eliminated from the conservation equations, prior to imposing the single-pole regime. As a consequence, our single-pole analysis turned out to differ from the existing literature.

The obtained equations of motion are found to differ from the known pole-dipole equations by the presence of a novel constraint on the particle spin tensor. With this constraint, the spin of the Dirac point particle turned out not to couple to the background curvature, leading to geodesic trajectories in torsionless spacetimes. In the presence of torsion, however, a geodesic deviation appears, but also a strong constraint, suggesting that the spin of the Dirac point particle might be zero. Being a consequence of our single-pole approximation, this unusual result has been checked by the explicit construction of the zero-size Dirac particle. To this purpose, a wave packet solution of the Dirac equation is considered in the limit of small size and wavelength. The expected single-pole behaviour has been verified, but also, the spin tensor has been found to disappear in this limit. In an attempt to explain this unusual behaviour, a relation between spin and orbital angular momentum has been discovered to hold in our wave packet example.

Before we close our exposition, let us comment on the possibility that the disappearance of spin in the zero-size limit might be a general property of all point particles. First, we notice that there is one more conserved current in the Dirac theory — the U(1) current  $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ . In our wave packet example, it is proportional to the wave vector  $k^\mu$ , and is related to the stress-energy  $\tau^{\mu\nu}$ . By a close inspection, we find that the following manifestly covariant relation holds:

$$\tau^{\mu\nu}j_\nu \propto j^\mu, \tag{5.1}$$

Its physical meaning is best seen in the rest frame where it reduces to  $\tau^{\alpha 0} = j^\alpha = 0$ . It tells us that the two currents are mutually proportional, i.e. that the energy and charge flow in the same direction. In the limit  $\lambda/\ell \rightarrow 0$ ,  $\ell \rightarrow 0$  it implies the constraint (4.5), and thus, explains why Dirac point particles have no spin.

The relations analogous to (5.1) might exist quite generally. There is nothing special about the statement that all the particle charges flow in the same direction. This is something one would expect to hold for any type of point-like matter. However, we need the relation (5.1) to hold for thick particles, as well. Only then, and only for Dirac matter, the consequence (4.5) has been derived. If (5.1) were a general property of the localized matter, the disappearance of spin in the zero-size limit might be a feature of all massive point particles. Indeed, when applied to other spins, the relation (5.1) implies

$$x^\alpha\tau^{(0\beta)} - x^\beta\tau^{(0\alpha)} = \sigma^{[\beta 0\alpha]} + x^{[\alpha}\partial_0\sigma^{00\beta]} + div.$$

After the integration, the divergence term vanishes, while l.h.s. and the second term on the r.h.s. give dipole coefficients of the order  $\mathcal{O}_1$ . In the single-pole regime, both disappear, and we end up with the vanishing spin.

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